1^{st} Lecture $E/\operatorname{Spec}\mathbb{C}$ EC $\Rightarrow$ $E^{an} \simeq \mathbb{C}/\Lambda$ (A)
torus

2^{nd} Lecture Smoothness $\Leftrightarrow$ geom. regular (B)

Today .) Normality & smoothness (A)

.) Algebraicity

.) j-invariant

$\Big\}$ (B)

Thursday Modular Curve / $\mathbb{C}$

(= moduli space of ECs)

After that Algebraic Theory

References Silverman § VI ECs / $\mathbb{C}$

§ III.1 Weierstraß equ's

§1 Normality of smooth curves

Prop $X/\operatorname{Spec} k$ smooth 1-dimensional. Then X is normal.

Proof By defn, means all $\mathcal{O}_{X,x}$ normal, i.e. DVRs or fields.

Last time: $X_{\bar{k}}$ normal (= regular for ≤ 1-dim) by smoothness.

Let $\bar{x} \in X_{\bar{k}} \longmapsto x \in X$. (Exists since $\bar{k} \otimes_k \kappa(x) \neq 0$.)

Then $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X_{\bar{k}},\bar{x}}$ since $\forall A$, $A \hookrightarrow \bar{k} \otimes_k A$.

(Two arguments may be summarized as "$\bar{k}/k$ faithfully flat".)

$\mathcal{O}_{X_{\bar{k}},\bar{x}}$ int dom $\Rightarrow$ $\mathcal{O}_{X,x}$ int dom.

If x is a generic point, then $\mathcal{O}_{X,x}$ is field (reduced + 0-dim + connected).

<u>Claim</u> Otherwise $\mathfrak{m}_x = (\pi)$ is principal.

Pf Equivalent $\mathfrak{m}_x$ be free (necessarily rk 1) over $\mathcal{O}_{X,x}$.

Enough: $\operatorname{Tor}_1(\mathfrak{m}_x, \kappa(x)) = 0$.

Can be shown after faithfully flat extn.

Then follows from fact that $\mathcal{O}_{X_{\bar{k}},\bar{x}}$ is DVR. $\square$

More refined versions of this argument show:

<u>Prop</u> (Stacks 00OF)

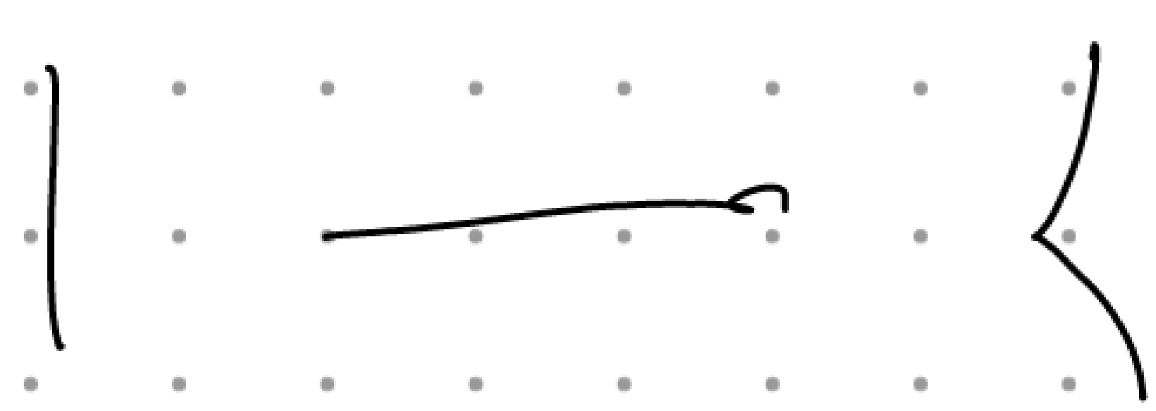
If $R \xrightarrow{\varphi} S$ is <u>flat</u> map of loc noeth rings s.th. $\varphi^{-1}(m_S) = m_R$ and S <u>regular</u>. Then R regular.

<u>Cor</u> (Stacks 038W)

If $X/_k$ smooth ($\Leftrightarrow X_{\bar{k}}$ regular), then X regular.

<u>Example</u> $\operatorname{Spec} k[t] \longrightarrow \operatorname{Spec} k[S,T]/\underbrace{T^2 - S^3}_{f}$

$$t^2, t^3 \longleftarrow\!\shortmid S, T$$



"cusp"

not flat, regularity does not descend.

Also seen from <u>Jacobian criterion</u>: $\left(\frac{\partial f}{\partial S}, \frac{\partial f}{\partial T}\right) = (-3S^2, 2T)$ vanishes at $(0,0)$.

(Smoothness $\Leftrightarrow$ rk Jac $f = 1$ everywhere)

Prop $f : X \longrightarrow Y$ non-constant map of proper smooth connected curves$/k$. Then f is <u>finite locally free</u>, i.e.

$f_* \mathcal{O}_X$ is finite loc free $\mathcal{O}_Y$-module.

Def $\deg(f) := \mathrm{rk}_{\mathcal{O}_Y} f_* \mathcal{O}_X$.

Pf ·) X connected $\Rightarrow$ $f(X)$ connected.

·) f non-constant $\Rightarrow$ $\eta_Y \in f(X)$, then necessarily

$$\eta_Y = f(\eta_X).$$

·) Since $\mathcal{O}_{X,\eta_X}$, $\mathcal{O}_{Y,\eta_Y}$ fields, X, Y integral schemes, implies $f_* \mathcal{O}_X$ torsion-free $\mathcal{O}_Y$-module.

·) Since X, Y proper, f is proper. Thus $f_* \mathcal{O}_X$ loc finite type $\mathcal{O}_Y$-module.

·) $\forall y \neq \eta_Y$, $\mathcal{O}_{Y,y}$ is $\overbrace{\text{DVR.}}^{\text{by smoothness}}$ Classification of fin gen modules over PIDs $\Rightarrow$ $(f_* \mathcal{O}_X)_y$ finite free $\mathcal{O}_{Y,y}$-module.

(Y noetherian $\Rightarrow$ $f_* \mathcal{O}_X$ even loc fin pres $\mathcal{O}_Y$-module)

·) Thus $f_* \mathcal{O}_X$ loc free over $\mathcal{O}_Y$. $\square$

Def $x \in X$ closed. Ramification index e_x } defined by
f as above Inertia degree f_x }

$f_x := [\kappa(x) : \kappa(f(x))]$, $\pi_{f(x)} \mathcal{O}_{X,x} = \pi_x^{e_x} \mathcal{O}_{X,x}$

$\pi_{f(x)}$, π_x uniformisers of resp. loc rings.

<u>Try yourself</u> $f : X \to Y$ non-constant as above

1) $(f_* \mathcal{O}_X)_{\eta_Y} \xrightarrow{\sim} \mathcal{O}_{X,\eta_X}$

In ptic, $\deg f = \dim_{\mathcal{O}_{Y,\eta_Y}} (f_* \mathcal{O}_X)_y = [\mathcal{O}_{X,\eta_X} : \mathcal{O}_{Y,\eta_Y}]$.

2) $\forall y \in Y$, $\deg f = \sum_{x \mapsto y} e_x \cdot f_x$

3) Have equiv of cats

$$\left\{ \begin{array}{c} \text{Prop. normal conn} \\ \text{curves} / \operatorname{Spec} k \\ + \text{non-constant} \\ \text{maps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Fin. gen. field ext } K/k \\ \text{of tr.deg. } 1 \end{array} \right\}$$

$$X \longmapsto \mathcal{O}_{X,\eta_X}$$

<u>1st way</u> Choose $k(t) \longrightarrow K$, take normalization of $\mathbb{P}^1_k$ in K.

<u>2nd way</u> $X := \{ \text{val rings } k \subseteq \mathcal{O}_{X,x} \subseteq K \}$

$\longmapsto K$

+ top s.th. $X \setminus Z$, Z finite set of $\mathcal{O}_{X,x} \neq K$, open
+ $\mathcal{O}_X(U) := \{ f \in K,\ f \in \mathcal{O}_{X,x}\ \forall x \in U \}$.

Def $X|k$ conn sm 1-dim. <u>Rational fctn on X</u> $\stackrel{=}{\text{def}}$

$$\mathcal{O}_{X,\eta} = \mathrm{Mor}(X, \mathbb{P}^1_k) = H^0(X, (\mathcal{O}_X \setminus \{0\})^{-1} \cdot \mathcal{O}_X)$$

From <u>valuative criterion of properness</u>:

$$\begin{tikzcd}
\mathrm{Spec}\, \mathcal{O}_{X,\eta} \arrow[r] \arrow[d] & \mathbb{P}^1_k \arrow[d] \\
\mathrm{Spec}\, \mathcal{O}_{X,x} \arrow[ur, dotted, "\exists!"] & \mathrm{Spec}\, k
\end{tikzcd}$$

Uses $\mathcal{O}_{X,x}$ are DVRs!

<u>Compare</u> $R := \left(k[S,T]/_{T^2 - S^3} \right)_{(0,0)}$ localization at max ideal.

$$\begin{tikzcd}
\mathrm{Frac}\, R & k[t] \arrow[l, hook', "\varphi"'] \\
R \arrow[u, hook] & k \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} \qquad \frac{T}{S} \mapsfrom t$$

has $\mathrm{Im}(\varphi) \not\subset R$.

<u>**Cor**</u> X conn sm proper 1-dim, $f : X \to \mathbb{P}^1_k$ rational function.

Then f takes every value $a \in \mathbb{P}^1_k$ equally often:

$$\sum_{x \mapsto a} e_x [\kappa(x) : \kappa(a)] = \deg f \quad \forall a.$$

§2 Meromorphic functions   Now $k = \mathbb{C}$.

Def X R.S. Merom fcts $\mathcal{M}(X)$ on $X \underset{\text{def}}{=}$

$f : X \to \mathbb{C} \cup \{\infty\}$ locally of form g/h, g, h holomorphic, $h \neq 0$ everywhere

$= \mathrm{Mor}(X, \mathbb{P}^1)$

By equiv of cats, if X is compact connected, $\mathcal{M}(X)$ is fin gen ext of $\mathbb{C}$ of tr. deg 1.

Cor (of alg. theory) X compact connected R.S., $f \in \mathcal{M}(X)$ non-const. Then f takes every $a \in \mathbb{C} \cup \{\infty\}$ equally often:

$$\sum_{x \in f^{-1}(a)} e_x \quad \text{indep of } a, \quad =: \deg f.$$

Analytic description of e_x: Pick chart $\mathbb{C} \supseteq U \xrightarrow[\sim]{\varphi} \varphi(U) \subseteq X$, $0 \mapsto x$, and write $(f \circ \varphi)(z) = \sum_{n \gg -\infty} c_n z^n$.

Set
$$e_x(f) := \begin{cases} \min\{n \neq 0 \mid c_n \neq 0\} & \text{if } f(x) \neq \infty \\ e_x(1/f) = -\min\{n \mid c_n \neq 0\} & f(x) = \infty. \end{cases}$$

Rmk $e_x(f)$ = valuation of f in DVR $\mathcal{O}_{X,x}$.

§3 Weierstraß $\wp$-function $\Lambda \subseteq \mathbb{C}$, $E = \mathbb{C}/\Lambda$.

<u>Aim</u> Determine field $\mathcal{M}(E)$ "field of elliptic fctn"

<u>Prop</u> (Liouville)

1) $H^0(E, \mathcal{O}_E) = \mathbb{C}$
2) $\nexists\ f \in \mathcal{M}(E)$ of degree 1
3) $\exists\ f$ of degree 2.

<u>Proof</u> 1) Holom fctn do not take maxima.

2) Such f would be isomorphism.

3) <u>Idea</u> $f(z) := \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2}$ Λ-invariant,
should descend to E.

<u>However</u>: Not convergent!

<u>Def</u> <u>Weierstraß $\wp$-fct for Λ</u>

$$\wp(z) :\overset{\text{def}}{=} \sum_{0 \neq \lambda \in \Lambda} \underbrace{\left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]}_{\in O(\lambda^{-3})} + \frac{1}{z^2}$$

Convergent, $\Leftarrow$ $\in O(\lambda^{-3})$
pole order 2 at $\lambda \in \Lambda$
Λ-invariant. $\square$

Note $\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} \in \mathcal{M}(E)$

also elliptic, pole order 3 at 0.

Thm $\mathcal{M}(E)$ is generated by $\wp, \wp'$. More precisely:

$\wp'$ is quadratic over $\mathbb{C}(\wp)$.

Rmk Shows that $E \xrightarrow{\wp} \mathbb{P}^1$ is (ramified) double cover.

Pf $\wp(-z) = \wp(z)$ <u>even</u> while $\wp'(-z) = -\wp'(z)$ <u>odd</u>

$\Rightarrow \wp' \notin \mathbb{C}(\wp)$.

Given $f \in \mathcal{M}(E)$,

$$f(z) = \frac{1}{2}\left(f(z) + f(-z)\right) + \frac{1}{2}\left(f(z) - f(-z)\right)$$

sum of even & odd function.

Since $\wp'$ odd is even, enough to show $\mathcal{M}(E)^{\text{even}} = \mathbb{C}(\wp)$.

Given f even w/ pole at $a \neq 0$,

$\left(\wp(z) - \wp(a)\right)^n f(z)$ has no pole at a for $n \gg 0$.

wlog 0 only (possible) pole of f.

f even $\to$ pole order even $\Rightarrow f = p(\wp)$ $p \in \mathbb{C}[t]$. $\square$

§4 Algebraicity

If $f(z) = \sum_{n \geq 0} a_n z^n$ holom near 0, then $a_n = \frac{1}{n!} f^{(n)}(0)$.

Apply to $\wp(z) - \frac{1}{z^2}$ (try yourself!) to obtain

$$\wp(z) = z^{-2} + 3G_4 z^2 + 5G_6 z^4 + \dots$$

$$\wp'(z) = -2z^{-3} + 6G_4 z + 20G_6 z^3 + \dots$$

w/ $G_k := \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}$.

$(\wp')^2$ even, pole order 6 at 0

$$\rightsquigarrow (\wp')^2 = 4\wp^3 - \underbrace{60G_4}_{g_2} \wp - \underbrace{140G_6}_{g_3}$$

<u>Thm</u> Let $C := V_+(ZY^2 - (4X^3 - g_2 Z^2 X - g_3 Z^3)) \subseteq \mathbb{P}^2_{\mathbb{C}}$.

Then $[\wp, \wp', 1] : E \xrightarrow{\sim} C^{an}$.

<u>Proof</u> Bijectivity of $E \to C(\mathbb{C})$: Try yourself.

Left to show: C smooth.

($X \xrightarrow{f} Y$ generically deg 1 morph of proper smooth connected curves / Spec k $\Rightarrow$ f iso.)

Only do chart $D_+(z)$ here:

Jacobian criterion for $V(\underbrace{Y^2 - (4X^3 - g_2 X - g_3)}_{g}) \subseteq \mathbb{A}^2_{\mathbb{C}}$.

(*)

$\frac{\partial g}{\partial Y} = 2Y$ only vanishes for $Y = 0$.

If $(x,0) \in C \cap D_+(z)$, then x zero of $4X^3 - g_2 X - g_3$

and $\left(\frac{\partial g}{\partial X}\right)(x,0) = 0 \iff x$ multiple zero.

Given zero x, $\exists!$ choice for y, namely $y = 0$.

<u>Claim</u> $y = \wp'(z) = 0 \iff z \in \left(\tfrac{1}{2}\Lambda/\Lambda\right) \setminus \{0\}$.

$\leadsto$ non-trivial 2-torsion.

<u>Pf</u> $2z = 0$, $z \neq 0 \implies \wp'(z) = -\wp'(-z)$

$= -\wp'(z) = 0$

Exhibits all 3 zeros of $\wp'$, which has $\deg \wp' = 3$. $\square$ Claim $\implies$ $\square$

<u>Upshot</u>

Every EC/Spec $\mathbb{C}$ is $\cong$ smooth cubic $\subseteq \mathbb{P}^2_{\mathbb{C}}$.

Will prove algebraically:

- ·) True for any field k.
- ·) Conversely, any smooth cubic E + point $e \in E(k)$ has a unique group structure s.th. $O = e$.

§5 j-invariant

Lem $\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda' \iff \Lambda' = a\Lambda$ some $a \in \mathbb{C}$.

Pf Any iso φ lifts to group iso of covers $\mathbb{C} \xrightarrow[\tilde{\varphi}]{\sim} \mathbb{C}$, is hence linear $\tilde{\varphi}(z) = az$. $\square$

But g_2, g_3 depend on Λ:

$$g_k(\Lambda) \sim \text{const} \cdot \sum_{0 \neq \lambda \in \Lambda} \lambda^{-2k}.$$

We see $g_k(a\Lambda) = a^{-2k} g_k(\Lambda) \qquad k = 2, 3$

Def discriminant $\Delta(\Lambda) := g_2(\Lambda)^3 - 27 g_3(\Lambda)^2$

j-invariant $j(\Lambda) := g_2(\Lambda)^3 / \Delta(\Lambda)$.

Lem $\Delta(\Lambda) \neq 0$.

Pf $\Delta(\Lambda)$ is discriminant of $4X^3 - g_2 X - g_3$, hence $\neq 0$ by smoothness of $\mathbb{C}/\Lambda$. (see ⊛). $\square$

Thm $j(E) := j(\Lambda)$ only depends on E, not choice of Λ.

I.e. for $E, E'/\mathbb{C}$, $\quad E \simeq E' \iff j(E) = j(E')$.